

# Slowly rotating, compact fluid sources embedded in Kerr empty space-time

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February 3, 2008

## Abstract

Spherically symmetric static fluid sources are endowed with rotation and embedded in Kerr empty space-time up to an including quadratic terms in an angular velocity parameter using Darmois junction conditions. Einstein's equation's for the system are developed in terms of linear ordinary differential equations. The boundary of the rotating source is expressed explicitly in terms of sinusoidal functions of the polar angle which differ somewhat according to whether an equation of state exists between internal density and supporting pressure.

Following the publication by Kerr [1] of the metric which describes analytically, the asymptotically flat, vacuum gravitational field outside a rotating source in terms of Einstein's field equations, there has been much discussion concerning the existence of possible interior solutions which match the exterior smoothly. In an important development Hartle [2] uses a second order perturbation technique to describe the slow rotation of equilibrium configurations of cold stars having constant angular velocity. Solutions of Einstein's equations are developed in terms of Legendre polynomials but the issue of matching the results to Kerr empty space-time is not addressed. In the case of non-equilibrium configurations Kegeles [3] has applied the method to Robertson-Walker dust sources up to the first order in angular velocity parameter although, the results are somewhat restrictive and are not suitable for application to sources supported by internal pressure. In a recent work the case of the Wahlquist [4] closed form interior was shown not to fit the Kerr exterior by Bradley *et al* [5]. Only for the important case of thin super-massive rotating discs, supported by internal pressure have analytic sources for the Kerr metric been found (Pichon and Lynden-Bell [6]). This has led to an 'embarrassing hiatus' according to Bradley *et al* [5] in the number of potential interior solutions available for matching which in turn has contributed to a lack in the development in the theory of differentially rotating fluid bodies in general relativity. Yet it is important to develop further the relativistic theory of rotation since it has considerable potential application in astrophysics, for example, in the description

of the gravitational collapse of rotating matter, quasars, or potential sources for gravitational radiation.

It is the aim here to focus upon one aspect of the problem of matching rotating interior solutions of Einstein's equations to the Kerr metric. In particular it is the intention to show how static perfect fluid sources endowed with rotation to an accuracy of second order terms in an angular velocity parameter  $q$  may be matched to the Kerr metric also expressed to an accuracy of quadratic terms in angular velocity parameter. In this sense the method employed is similar to that of Bradley *et al* [5]. Moreover, the results given here apply to fluid interiors having variable interior angular velocity and so to that extent generalise the discussion of Hartle [2]. Indeed the problem considered here is essentially that of matching the generalised Hartle rotating interiors to Kerr empty space-time although the approach here differs significantly. The matching problem is addressed by means of the application of Darmois [7] junction conditions which are discussed widely in the literature for example, Misner et al [8], Mars & Senovilla [9], Stephani [10], Hernandez-Pastora et al [11] and Bonnor and Vickers [12]. In the Darmois approach it is necessary that the components of the metric tensor, and also the extrinsic curvature for the Kerr exterior and the interior source are continuous at the boundary surface.

In the following Einstein's equations for a perfect fluid source will be expressed in the form:

$$G_b^a = -8\pi T_b^a \quad , \quad T_b^a = (\rho + p) u^a u_b - \delta_b^a p \quad , \quad (1)$$

where  $\rho$ ,  $p$  are the respective rotating source density and supporting internal pressure and  $u^a$  are the components of the velocity four-vector with the property that  $u^a u_a = 1$ .

## 1 The fluid source and Darmois junction conditions with empty space-time

Consider the well known spherically static perfect fluid source which is represented here by the metric:

$$d\sigma_S^2 = e^{2\lambda} d\eta^2 - e^{2\mu} d\xi^2 - \xi^2 d\theta^2 - \xi^2 \sin^2 \theta d\phi^2 \quad (2)$$

where  $\lambda = \lambda(\xi)$  and  $\mu = \mu(\xi)$  such that

$$\lambda_{\xi\xi} = -\lambda_{\xi}^2 + \mu_{\xi} \lambda_{\xi} + \frac{\lambda_{\xi} + \mu_{\xi}}{\xi} - \frac{e^{2\mu}}{\xi^2} + \frac{1}{\xi^2} \quad (3)$$

and a suffix indicates a derivative. The respective supporting pressure and density of the static cases are:

$$8\pi p_S = \frac{2e^{-2\mu} \lambda_{\xi}}{\xi} + \frac{e^{-2\mu}}{\xi^2} - \frac{1}{\xi^2} \quad (4)$$

and

$$8\pi\rho_S = \frac{2e^{-2\mu}\mu_\xi}{\xi} - \frac{e^{-2\mu}}{\xi^2} + \frac{1}{\xi^2} \quad (5)$$

In the following this source will be endowed with rotation up to and including quadratic terms in angular velocity parameter  $q$ . It is assumed that the resulting physical system may be described by means of the metric

$$\begin{aligned} d\sigma^2 = & e^{2\lambda} (1 + Qq^2) d\eta^2 - e^{2\mu} (1 + Uq^2) d\xi^2 - \xi^2 (1 + Vq^2) d\theta^2 \\ & - \xi^2 (1 + Wq^2) \sin^2 \theta d\phi^2 - 2X\xi^2 q \sin^2 \theta d\phi d\eta \end{aligned} \quad (6)$$

where  $U, V, W$  and  $Q$  are each functions of  $\xi$  and  $\theta$  whilst  $X$  is a function of  $\xi$  alone. This particular gauge has been chosen since it is the intention to match this interior source to empty space-time using the well known Kerr metric expressed in Boyer and Lindquist coordinates at a boundary between source and empty space-time given by:

$$\xi_b = \xi_0 + q^2 f(\theta) \quad (7)$$

where  $f = f(\theta)$  and  $\xi_0$  is constant. All expressions will be accurate up to and including quadratic terms in  $q$  so that the Kerr metric may be written as:

$$d\sigma^2 = \Psi_{44} d\eta^2 - 2\Psi_{43} q \sin^2 \theta d\phi d\eta - \Psi_{11} d\xi^2 - \Psi_{22} d\theta^2 - \Psi_{33} \sin^2 \theta d\phi^2 \quad (8)$$

where:

$$\begin{aligned} \Psi_{44} &= 1 - \frac{2m}{\xi} \left( 1 - \frac{q^2 \cos^2 \theta}{\xi^2} \right) \\ \Psi_{43} &= \frac{2m}{\xi} \\ \Psi_{11} &= \frac{1}{1 - \frac{2m}{\xi}} \left[ 1 + \frac{q^2 \cos^2 \theta}{\xi^2} - \frac{q^2}{\xi^2 \left( 1 - \frac{2m}{\xi} \right)} \right] \\ \Psi_{22} &= \xi^2 + q^2 \cos^2 \theta \\ \Psi_{33} &= \frac{2mq^2 \sin^2 \theta}{\xi} + q^2 + \xi^2 \end{aligned} \quad (9)$$

Darmois junction conditions are employed at the boundary so that both the metric tensor  $g_{ab}$  and the extrinsic curvature  $K_{ab}$  must be continuous on the boundary surface (7). Using the suffix  $b$  to indicate evaluation at  $\xi = \xi_b$  and the suffix 0 to indicate evaluation at  $\xi = \xi_0$  it is straight forward to show that the following conditions on  $\lambda, \mu, U, V, W, Q$  and  $X$  and derivatives with respect to  $\xi$  hold at the boundary:

$$\begin{aligned}
\{e^{2\lambda}\}_0 &= \{e^{-2\mu}\}_0 = 1 - \frac{2m}{\xi_0} \equiv K \\
\{\lambda_\xi\}_0 &= \frac{1-K}{2K\xi_0} \\
U_b &= \frac{K \cos^2 \theta - 1}{K\xi_0^2} + 2f\{\mu_\xi\}_0 + \frac{f(K-1)}{K\xi_0} \\
V_b &= \frac{\cos^2 \theta}{\xi_0^2} \\
\{V_\xi\}_b &= \frac{4f\{\mu_\xi\}_0}{\xi_0} - \frac{2 \cos^2 \theta}{\xi_0^3} \\
W_b &= \frac{1 + (1-K) \sin^2 \theta}{\xi_0^2} \\
\{W_\xi\}_b &= \frac{3(K-1) \sin^2 \theta - 2}{\xi_0^3} + \frac{4f\{\mu_\xi\}_0}{\xi_0} \\
Q_b &= \frac{(1-K) \cos^2 \theta}{K\xi_0^2} \\
\{Q_\xi\}_b &= \frac{(2K^2 - K - 1) \cos^2 \theta}{\xi_0^3 K^2} + \frac{f(1-3K)\{\mu_\xi\}_0}{\xi_0 K} + \frac{f(K^2-1)}{2\xi_0^2 K^2} \\
X_b &= \frac{1-K}{\xi_0^2} \\
\{X_\xi\}_b &= \frac{3(K-1)}{\xi_0^3}
\end{aligned} \tag{10}$$

For future convience it is useful to define the functions  $H(\xi, \theta)$  and  $h(\xi)$  such that:

$$\begin{aligned}
H(\xi, \theta) &= Q(\xi, \theta) + \sin^2 \theta \xi^2 X^2 e^{-2\lambda} \\
X_\xi &= \frac{h(\xi) e^{\lambda+\mu}}{\xi^4}
\end{aligned} \tag{11}$$

so:

$$\begin{aligned}
H_b &= \frac{(1-K)}{K\xi_0^2} (1 - K \sin^2 \theta) \\
\{H_\xi\}_b &= \frac{3 \sin^2 \theta (1-K)}{\xi_0^3} + \frac{(2K^2 - K - 1)}{K^2 \xi_0^3} + \frac{f(K^2-1)}{2\xi_0^2 K^2} \\
&\quad + \frac{f\{\mu_\xi\}_0 (1-3K)}{\xi_0 K} \\
h_b &= 3\xi_0 (K-1)
\end{aligned} \tag{12}$$

## 2 Einstein's equations for the rotating source

There are three non-trivial Einstein equations (1) which need to be solved for a fluid source in the gauge (6). In the first the condition  $T_2^1 = 0 = T_1^2$  gives rise to:

$$-U_\theta \lambda_\xi + H_\theta \lambda_\xi + W_{\theta\xi} - \frac{U_\theta}{\xi} + H_{\theta\xi} - \frac{H_\theta}{\xi} + \frac{\cos \theta (W_\xi - V_\xi)}{\sin \theta} = 0 \quad (13)$$

whilst  $T_1^1 = T_2^2$  becomes:

$$\begin{aligned} & \frac{e^{-2\mu}}{2} \left\{ -V_\xi \lambda_\xi - U_\xi \lambda_\xi + 2H_\xi \lambda_\xi + W_{\xi\xi} + \frac{W_\xi}{\xi} - \mu_\xi W_\xi \right. \\ & \left. - \frac{V_\xi}{\xi} - \frac{U_\xi}{\xi} + H_{\xi\xi} - \frac{H_\xi}{\xi} - \mu_\xi H_\xi \right\} \\ & + \frac{\cos \theta (U_\theta + V_\theta - 2W_\theta)}{2\xi^2 \sin \theta} - \frac{H_{\theta\theta} + W_{\theta\theta}}{2\xi^2} + \frac{U - V}{\xi^2} \\ & - \frac{h^2 \sin^2 \theta}{2\xi^6} \\ & = 0 \end{aligned} \quad (14)$$

The final equation is  $(T_3^3 + p)(T_4^4 + p) - T_4^3 T_3^4 = 0$  which is explicitly:

$$\begin{aligned} & e^{-2\mu} (\lambda_\xi + \mu_\xi) \left\{ -W_{\xi\xi} - \lambda_\xi W_\xi - \frac{2W_\xi}{\xi} + \mu_\xi W_\xi + V_{\xi\xi} \right. \\ & \left. + \lambda_\xi V_\xi + \frac{2V_\xi}{\xi} - \mu_\xi V_\xi \right\} - \frac{\cos \theta (H_\theta + U_\theta) (\lambda_\xi + \mu_\xi)}{\xi^2 \sin \theta} \\ & + \frac{(H_{\theta\theta} + U_{\theta\theta}) (\lambda_\xi + \mu_\xi)}{\xi^2} - \sin^2 \theta \left( \frac{h_\xi^2}{4\xi^5} + \frac{(\lambda_\xi + \mu_\xi) h^2}{\xi^6} \right) \\ & = 0 \end{aligned} \quad (15)$$

An important differential consequence of these equations, since it is independent of any second derivatives with respect to  $\xi$  is the relation:

$$\begin{aligned} & \frac{e^{-2\mu} \sin \theta}{\cos \theta} (U_\theta - H_\theta) \xi^2 \lambda_\xi (\lambda_\xi - \mu_\xi) (\lambda_\xi + \mu_\xi) \\ & + e^{-2\mu} (V_\xi - W_\xi) (\xi^2 \lambda_\xi - \mu_\xi \xi^2 + 2\xi) (\lambda_\xi + \mu_\xi) \\ & + \frac{e^{-2\mu} \lambda_\xi \sin \theta}{\cos \theta} (\xi^2 V_{\theta\xi} - \xi U_\theta - \xi^2 H_{\theta\xi} + \xi H_\theta) (\lambda_\xi + \mu_\xi) \\ & + \frac{\sin \theta}{\cos \theta} (W_{\theta\theta\theta} - 2W_\theta + 3V_\theta + H_{\theta\theta\theta} - H_\theta) (\lambda_\xi + \mu_\xi) \\ & + \frac{e^{-2\mu} (\lambda_\xi + \mu_\xi) \sin \theta}{\cos \theta} \{ \mu_\xi (\xi^2 W_{\theta\xi} - \xi U_\theta + \xi^2 H_{\theta\xi} + \xi H_\theta) \\ & - \xi W_{\theta\xi} + \xi V_{\theta\xi} + 2H_\theta \} + \frac{\cos \theta}{\sin \theta} (-2W_\theta + V_\theta - H_\theta) (\lambda_\xi + \mu_\xi) \\ & + (2W_{\theta\theta} - V_{\theta\theta} + H_{\theta\theta}) (\lambda_\xi + \mu_\xi) + \sin^2 \theta \left( \frac{(\lambda_\xi + \mu_\xi) h^2}{\xi^4} - \frac{h_\xi^2}{4\xi^3} \right) \\ & = 0 \end{aligned} \quad (16)$$

The expression for the supporting internal pressure is:

$$\begin{aligned}
8\pi p = & 8\pi p_S + e^{-2\mu} q^2 \left( \frac{W_\xi \lambda_\xi}{2} + \frac{V_\xi \lambda_\xi}{2} - \frac{2U \lambda_\xi}{\xi} + \frac{W_\xi}{2\xi} + \frac{V_\xi}{2\xi} - \frac{U}{\xi^2} + \frac{H_\xi}{\xi} \right) \\
& + \frac{q^2}{2\xi^2} (W_{\theta\theta} + 2V + H_{\theta\theta}) \\
& + \frac{q^2 \cos \theta (2W_\theta - V_\theta + H_\theta)}{2\xi^2 \sin \theta} + \frac{h^2 q^2 \sin^2 \theta}{4\xi^6}
\end{aligned} \tag{17}$$

and the fluid density is:

$$\begin{aligned}
8\pi \rho = & 8\pi p_S + e^{-2\mu} q^2 \left\{ \frac{W_\xi \lambda_\xi}{2} - \frac{V_\xi \lambda_\xi}{2} - \frac{W_\xi}{2\xi} - V_{\xi\xi} - \frac{5V_\xi}{2\xi} + \mu_\xi V_\xi \right. \\
& + \left. \frac{U_\xi}{\xi} - \frac{2\mu_\xi U}{\xi} + \frac{U}{\xi^2} \right\} - \frac{q^2}{2\xi^2} (W_{\theta\theta} + 2V + 2U_{\theta\theta} + H_{\theta\theta}) \\
& + \frac{q^2 \cos \theta (-2W_\theta + V_\theta + H_\theta)}{2\xi^2 \sin \theta} + \frac{h^2 q^2 \sin^2 \theta}{4\xi^6}
\end{aligned} \tag{18}$$

The angular velocity may be written as:

$$L(\xi, \eta) \equiv \frac{u^3}{u^4} = \frac{T_4^3}{T_4^4 + p} = \frac{qh_\xi e^{\lambda+\mu}}{4(\lambda_\xi + \mu_\xi)\xi^3} - qX \tag{19}$$

### 3 Description of the boundary

Application of the boundary conditions (10) and (12) show that equation (13) is satisfied identically whilst (17) gives:

$$8\pi p_b = 0 \tag{20}$$

as expected.

Whenever  $\lambda_\xi + \mu_\xi \neq 0$ , so that from the second of (10)

$$J = 2\xi_0 \{\mu_\xi\}_0 K - K + 1 \neq 0 \tag{21}$$

then equation (16) may be used to define  $f(\theta)$  since at the boundary it reduces to:

$$\begin{aligned}
& 2 \cos \theta \sin \theta K J \left( 4\xi_0 \{\mu_\xi\}_0 K (K - 1) + 7K^2 - 10K - 1 \right) \\
& - \frac{\xi_0 f_\theta J}{2K} \left( 2\xi_0 \{\mu_\xi\}_0 K + K - 1 \right) \left( 2\xi_0 \{\mu_\xi\}_0 K (K + 1) - 3K^2 + 4K - 1 \right) \\
& - \cos \theta \sin \theta K^2 \{h_\xi^2\}_b \\
= & 0
\end{aligned} \tag{22}$$

Thus in such cases it follows that:

$$f(\theta) = a_0 + a_1 \sin^2 \theta \tag{23}$$

where  $a_0$  is an arbitrary constant and  $a_1$  is found using (22).

However the case  $\{\lambda_\xi + \mu_\xi\}_0 = 0$  is also of importance. From equation (16) it follows that  $\{h_\xi\}_b = 0$  and so the value of the angular velocity  $L(\xi_b, \eta)$  at the boundary surface must be determined by careful analysis of the limiting behaviour of (19) at  $\xi = \xi_b$ . Moreover since this case corresponds to the condition  $J = 0$  then the second of (12) and (5) imply that the surface density for the static solution must satisfy  $\{8\pi\rho_S\}_0 = 0$ . This will invariably be the case in physical situations where an equation of state of the type  $p = F(\rho)$  is additionally imposed on equations (13) to (15) to close the system of Einstein's equations. For such cases the condition  $\{8\pi\rho\}_b = 0$  also gives:

$$\begin{aligned} & f \left( \frac{2K^2 \{\mu_{\xi\xi}\}_0}{\xi_0} - \frac{10K^2 \{\mu_\xi\}_0}{\xi_0^2} + \frac{K^2 + K - 2}{\xi_0^3} \right) \\ & + K^2 \left( -\{V_{\xi\xi}\}_0 + \frac{\{U_\xi\}_0}{\xi_0} + \frac{8\cos^2\theta}{\xi_0^4} \right) \\ & + f_{\theta\theta} \left( -\frac{2K \{\mu_\xi\}_0}{\xi_0^2} + \frac{1-K}{\xi_0^3} \right) - \frac{1+K}{\xi_0^4} \\ & = 0 \end{aligned} \tag{24}$$

This equation may be used to define the equation of the boundary  $f = f(\theta)$  whenever  $\{U_\xi\}_0$  and  $\{V_{\xi\xi}\}_0$  are known. Clearly whenever  $\{U_\xi\}_0$  and  $\{V_{\xi\xi}\}_0$  depend purely on terms in  $\sin^2\theta$  then  $f(\theta)$  will have a solution of the form (23) and  $a_0$  and  $a_1$  would be found using (24).

## 4 Generating solutions of Einstein's equations

Consider now the condition  $\{\lambda_\xi + \mu_\xi\}_0 \neq 0$  and also the condition  $\{\lambda_\xi + \mu_\xi\}_0 = 0$  provided that  $\{U_\xi\}_0$  and  $\{V_{\xi\xi}\}_0$  depend purely on terms in  $\sin^2\theta$ . The case when  $\{U_\xi\}_0$  and  $\{V_{\xi\xi}\}_0$  depend on higher harmonics is discussed in the next section.

It follows that  $f(\theta)$  has the form (23) and for such cases solutions of Einstein's equations satisfying the boundary Darmois boundary conditions may be found by writing:

$$\begin{aligned} U(\xi, \theta) &= G_0(\xi) + G_1(\xi) \sin^2\theta \\ V(\xi, \theta) &= G_2(\xi) + G_3(\xi) \sin^2\theta \\ W(\xi, \theta) &= G_2(\xi) + G_5(\xi) \sin^2\theta \\ H(\xi, \theta) &= G_6(\xi) + G_7(\xi) \sin^2\theta \end{aligned} \tag{25}$$

The equations (13), (14) and (15) then give rise to four independent equations for  $G_i(\xi)$  as follows:

$$2G_7\lambda_\xi - 2G_1\lambda_\xi - \frac{2G_7}{\xi} - \frac{2G_1}{\xi} + 2G_{7\xi} + 3G_{5\xi} - G_{3\xi} = 0 \tag{26}$$

$$\begin{aligned}
& 2G_{7\xi}\lambda_\xi - G_{3\xi}\lambda_\xi - G_{1\xi}\lambda_\xi - \frac{G_{7\xi}}{\xi} + \frac{G_{5\xi}}{\xi} - \frac{G_{3\xi}}{\xi} - \frac{G_{1\xi}}{\xi} \\
& + \frac{4G_7 e^{2\mu}}{\xi^2} + \frac{8G_5 e^{2\mu}}{\xi^2} - \frac{4G_3 e^{2\mu}}{\xi^2} - \frac{h^2 e^{2\mu}}{\xi^6} \\
& - G_{7\xi}\mu_\xi - G_{5\xi}\mu_\xi + G_{7\xi\xi} + G_{5\xi\xi} \\
= & 0
\end{aligned} \tag{27}$$

$$\begin{aligned}
& 2G_7 \left( \lambda_\xi^2 - \frac{\lambda_\xi}{\xi} - \mu_\xi \lambda_\xi - \frac{\mu_\xi}{\xi} + \frac{6e^{2\mu}}{\xi^2} - \frac{2}{\xi^2} \right) + \\
& 2G_1 \left( -\lambda_\xi^2 + \frac{\lambda_\xi}{\xi} + \mu_\xi \lambda_\xi + \frac{\mu_\xi}{\xi} \right) + \frac{h_\xi^2 e^{2\mu}}{4\xi^5(\lambda_\xi + \mu_\xi)} \\
& + 2G_{7\xi}(\lambda_\xi - \mu_\xi) + G_{5\xi} \left( \lambda_\xi + \frac{4}{\xi} - 3\mu_\xi \right) \\
& + G_{3\xi} \left( -3\lambda_\xi - \frac{4}{\xi} + \mu_\xi \right) + \frac{16G_5 e^{2\mu}}{\xi^2} - \frac{8G_3 e^{2\mu}}{\xi^2} - \frac{h^2 e^{2\mu}}{\xi^6} \\
= & 0
\end{aligned} \tag{28}$$

$$\begin{aligned}
& -2G_{6\xi}\lambda_\xi + G_{2\xi}\lambda_\xi + G_{0\xi}\lambda_\xi + \frac{G_{6\xi}}{\xi} + \frac{G_{0\xi}}{\xi} \\
& + \frac{2(G_7 + 3G_5 - G_3 + G_2 - G_1 - G_0)e^{2\mu}}{\xi^2} \\
& + G_{6\xi}\mu_\xi + G_{2\xi}\mu_\xi - G_{6\xi\xi} - G_{2\xi\xi} \\
= & 0
\end{aligned} \tag{29}$$

The first three contain only  $G_1(\xi)$ ,  $G_3(\xi)$ ,  $G_5(\xi)$  and  $G_7(\xi)$  whilst the fourth contains each of the unknown  $G_i(\xi)$ . This under-determined set may be closed with the aid of an imposed equation of state as discussed in the previous section.

The boundary conditions (10) and (12) together with (25) then give rise to the following:

$$\begin{aligned}
G_0(\xi_0) &= \frac{K-1}{K} \left( \frac{1}{\xi_0^2} + \frac{a_0}{\xi_0} \right) + 2a_0 \{\mu_\xi\}_0 \\
G_1(\xi_0) &= 2a_1 \{\mu_\xi\}_0 - \frac{1}{\xi_0^2} - \frac{a_1(1-K)}{\xi_0 K} \\
G_2(\xi_0) &= \frac{1}{\xi_0^2} \\
G_3(\xi_0) &= -\frac{1}{\xi_0^2} \\
G_5(\xi_0) &= \frac{1-K}{\xi_0^2} \\
G_6(\xi_0) &= \frac{1-K}{K\xi_0^2} \\
G_7(\xi_0) &= \frac{K-1}{\xi_0^2}
\end{aligned} \tag{30}$$



and:

$$\begin{aligned}
G_{2\xi}(\xi_0) &= \frac{4\{\mu_\xi\}_0 a_0}{\xi_0} - \frac{2}{\xi_0^3} \\
G_{3\xi}(\xi_0) &= \frac{4\{\mu_\xi\}_0 a_1}{\xi_0} + \frac{2}{\xi_0^3} \\
G_{5\xi}(\xi_0) &= \frac{4\{\mu_\xi\}_0 a_1}{\xi_0} + \frac{3(K-1)}{\xi_0^3} \\
G_{6\xi}(\xi_0) &= \frac{a_0\{\mu_\xi\}_0}{\xi_0 K} - \frac{1}{\xi_0^3 K} - \frac{a_0}{2\xi_0^2 K^2} - \frac{1}{\xi_0^3 K^2} \\
&\quad - \frac{3a_0\{\mu_\xi\}_0}{\xi_0} + \frac{a_0}{2\xi_0^2} + \frac{2}{\xi_0^3} \\
G_{7\xi}(\xi_0) &= -\frac{3K}{\xi_0^3} + \frac{a_1\{\mu_\xi\}_0}{\xi_0 K} - \frac{a_1}{2\xi_0^2 K^2} - \frac{3a_1\{\mu_\xi\}_0}{\xi_0} \\
&\quad + \frac{a_1}{2\xi_0^2} + \frac{3}{\xi_0^3}
\end{aligned} \tag{31}$$

## 5 Discussion and conclusion

In the analysis above it has been shown how the problem of determining slowly rotating sources embedded in Kerr empty space-time has reduced to the problem of determining solutions of the system of ordinary differential equations (26) to (29) subject to the conditions (30) and (31). Whilst these equations are under-determined they may be supplemented by an equation of state to make unique solution possible. Clearly the solution of the equations is straight forward enough although analytic solution in closed form for any known  $\lambda$  and  $\mu$  seem not to be possible.

The most appropriate techniques for solution are either numerical or alternatively a Taylor's series solution approach for  $G_i(\xi)$  about  $\xi = \xi_0$ . Two variations on the series method are worthy of note. In the first one may assume that both  $\lambda$  and  $\mu$  are known, for example, the Schwarzschild interior solution may be specified, and then take the system (26) to (29) subject to the conditions (30) and (31) as a linear set of equations for  $G_i(\xi)$ . In the second one may assume that neither  $\lambda$  and  $\mu$  are known, then impose an appropriate equation of state and also incorporate (3) so as to develop a series solution for  $\lambda$  and  $\mu$ .

It should be noted that solution of the equations (26) to (29) does not give the most general solution of the problem of determining stationary rotating fluid sources which match Kerr empty space-time. More general solutions may be obtained by appending terms of the type:

$$\sum_{k=2}^{\infty} Y_k(\xi) \sin^{2k} \theta \tag{32}$$

to each the substitutions (25) for  $U$ ,  $V$ ,  $W$  and  $H$  and then to generate a further sequence of linear ordinary differential equations for  $Y_k(\xi)$  similar to

(26) to (29). In the absence of an imposed equation of state then the resulting equations would be solved subject to the conditions  $Y_k(\xi_0) = 0$  and derivative  $Y_{k_\xi}(\xi_0) = 0$  for each value of  $k$ . The equation of the boundary in case is (23). When an equation of state is imposed then the equation of the boundary is determined from (24) and may contain higher harmonic component  $\sin^{2k} \theta$ ,  $k \geq 2$  depending on the nature of the boundary expressions for  $\{U_\xi\}_0$  and  $\{V_{\xi\xi}\}_0$ . These expressions will in turn determine the boundary conditions which should be imposed on  $Y_k(\xi_0)$  and the derivatives  $Y_{k_\xi}(\xi_0)$ .

The solution techniques outlined here are of course standard and are therefore not pursued further. The relevance of the analysis to, for example, determining potential sources for gravitational radiation or determining more rapidly fluid sources matching empty space-time will be examined in future research.

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